# Lecture 28 Summary

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## 1 $^3He$ as a Prototypical Fermi Liquid and "Exotic" Superconductor

$^4He$ is a well-know Bosonic atom that undergoes a superfluid transition at about 2.2 K in to a Bose-Einstein-condensate-like state. It's small mass and lack of chemistry makes it ideal for showing Bosonic quantum fluid properties.
$^3He$ on the other hand has one un-paired neutron in the nucleus and a net spin-1/2 nucleus, making it a Fermion. It is even lighter than $^4He$, making for even more interesting quantum effects. At low temperatures it condenses into a fluid which show prototypical Landau Fermi Liquid properties. Eventually, at about 2.8 mK, it makes a transition in to a superfluid state that resembles BCS superconductivity, although without the electrical charge.
Our objective is to review the Fermi liquid properties and then discuss the superfluid state, making contact with BCS theory whenever possible.

## 2 The Landau Fermi Liquid Theory of $^3He$

At low temperatures (3 to 100 mK) $^3He$ is a degenerate Fermi liquid. The Fermi energy $E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$ is equal to just 0.5 meV, as opposed to about 10 eV in good metals. The corresponding Fermi temperature is just 4.9 K, and the Fermi wavenumber is 0.78 $^{-1}$.

The heat capacity above $T_c$ is observed to be linear in temperature, just like the electronic heat capacity in metals, but with a slope 3 times bigger than theory, $C = \frac{\pi^2 k_B^2 NT}{2E_F}$.

The magnetic properties are dominated by the un-paired nuclear spin, which has a moment $\mu_N = 5.4 \times 10^{-4}\mu_B$, where $\mu_B$ is the Bohr magneton. There is a temperature-independent paramagnetic susceptibility at temperatures below 1 K, analogous to Pauli paramagnetism for the electron gas in a metal.

Landau devleoped a version of Fermi liquid theory in 1956 to explain the low-temperature properties of $^3He$. He started with the non-interacting gas and treated the interactions as a perturbation. He treated all particles as delocalized, and used momentum eigenstates, $\psi \sim \frac{1}{\sqrt{V}} e^{-i\vec{k}\cdot\vec{r}} \chi$ similar to BCS. After turning on the interactions it is assumed that there is a 1:1 correspondence between the states of the system, and that the interactions simply change the energies of these states. By adiabatic continuation, the wavefunctions smoothly evolve as the interactions are turned on. The Hamiltonian has the form, $H = \sum_{i=1}^{N} -\frac{\hbar^2}{2m} \nabla_i^2 + \frac{\lambda}{2} \sum_{i \neq j} V(\vec{r}_i - \vec{r}_j)$. Here $\lambda$ is tuned from 0 (no interactions) to 1 (full interactions) adiabatically. It is assumed that the atoms interact with each other purely by means of two-body interactions.

The interactions between particles can be written in terms of a small number of parameters as follows. Write the interaction energy as, $E_{int} = f_1(\vec{k}, \vec{k}') + f_2(\vec{k}, \vec{k}')\vec{S} \cdot \vec{S}'$. The first term depends only on the momentum of the atoms, which is typically very close to $k_F$. The second term is the spin-dependent part, which depends on the vector dot product of the two spins. Using the angle $\Theta$ between the directions $\vec{k}$ and $\vec{k}'$, Landau defined his dimensionless parameters $F$ and $G$ as follows,
$F(\Theta) = N(E_F) f_1(\Theta) = \sum_{n=0}^{\infty} F_n P_n(\cos(\Theta)) = F_0 + F_1 \cos\Theta + ...$, and
$G(\Theta) = N(E_F) f_2(\Theta) = \sum_{n=0}^{\infty} G_n P_n(\cos(\Theta)) = G_0 + G_1 \cos\Theta + ...$,
where $N(E_F)$ is the density of states at the Fermi energy and one has expanded in a set of Legendre polynomials over the spherical Fermi surface. It turns out that most of the normal state physical properties can be understood in terms of just $F_0$, $F_1$ and $G_0$. For example, it is found that the mass of the quasiparticle is enhanced as $m_3^* = m_3(1 + \frac{1}{3}F_1)$, which turns out to be about a factor of 3. This explains the enhanced slope of the heat capacity vs. temperature. It is also found that the magnetic susceptibility is enhanced as $\chi = \frac{m_3^*}{m_3} \frac{\chi^{ideal}}{1+\frac{1}{4}G_0}$ because $G_0 < 0$.

The Landau parameters for $^3He$ as a function of pressure are,

| | | $Pressure(bar)$ | |
|---|---|---|---|
| | 0 | 15 | 30 |
| $F_0$ | 10 | 46 | 82 |
| $F_1$ | 6 | 11 | 14.6 |
| $G_0$ | $-2.69$ | $-2.92$ | $-2.95$ |

Note that the solid phase sets in just above a pressure of 30 bar. Note that the spin-dependent terms are very weakly dependent on pressure.

## 3 Superfluid Properties of $^3He$

$^3He$ has three distinct superfluid phases, A, B and A1. In zero magnetic field it will condense in to either the A or B phase, depending on the pressure. With increasing field the B phase is reduced and the A phase takes over. The B phase supports persistent angular momentum states while the A phase does not. There is a jump in specific heat upon going from the normal phase to the A phase in zero field, reminiscent of the BCS transition. In non-zero field there are two second order phase transitions as the system goes from the normal phase to the A1 phase to the A phase. The transition from A to B phase is first order, with latent heat and hysterisis.

The short-range Lennard-Jones ($1/r^{12}$) repulsion is stronger than that in electron-electron interactions, such that the spin-singlet s-wave pairing channel is suppressed. The $l = 1$ angular momentum pairing state with spin-triplet pairing is favored. The large paramagnetic susceptibility of the $^3He$ atoms favors the S=1 pairing. For the pair, the angular momentum vector and the spin vector can, in general, point in different directions, making for many possible pairing states.

# 4 Pairing in $^3He$

Leggett found that the pairing interaction between $^3He$ atoms can be written as,
$V_{k,k'} \approx \frac{1}{N(E_F)} \frac{G_0}{1+\frac{1}{4}G_0} \overrightarrow{S} \cdot \overrightarrow{S}'$. We saw above that $G_0 \approx -3$, hence the pairing interaction favors ferromagnetic alignment of the atom nuclear spins.

The pairing interaction is now more complicated because we have to keep more careful track of the spin,
$H_{int} = \sum_{k,k';\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}(k,k') c^+_{k',\alpha} c^+_{-k',\beta} c_{-k,\gamma} c_{k,\delta}$. This is a generalization of the BCS pairing Hamiltonian. It is again assumed that Cooper pairs have zero net momentum in the ground state.

One can define a BCS-like order parameter as follows,
$F_{\alpha\beta}(k) = \langle c_{-k,\alpha} c_{k,\beta} \rangle$
$= \begin{pmatrix} \langle c_{-k,\uparrow} c_{k,\uparrow} \rangle & \langle c_{-k,\uparrow} c_{k,\downarrow} \rangle \\ \langle c_{-k,\downarrow} c_{k,\uparrow} \rangle & \langle c_{-k,\downarrow} c_{k,\downarrow} \rangle \end{pmatrix}$
giving pairing amplitude in 4 different channels.

The BCS gap equation is now,
$\Delta_{\alpha\beta}(k) = \sum_{k',\gamma\delta}(k,k') V_{\alpha\beta\gamma\delta}(k,k') \langle c_{-k',\gamma} c_{k',\delta} \rangle$.
The final gap equation can be written as,
$\begin{pmatrix} \Delta_{\uparrow\uparrow}(k) & \Delta_{\uparrow\downarrow}(k) \\ \Delta_{\downarrow\uparrow}(k) & \Delta_{\downarrow\downarrow}(k) \end{pmatrix} = i \left( \Delta_k I_{2\times2} + \overrightarrow{d}(k) \cdot \overrightarrow{\sigma} \right) \sigma_y$, where $\overrightarrow{\sigma}$ is the vector of Pauli spin matrices and $\overrightarrow{d}(k)$ is a vector order parameter for the spin triplet pairing state.

The quasiparticle excitation spectrum can be written as,

$E_k = \sqrt{(\epsilon_k - \mu)^2 + |\overrightarrow{d}(k)|^2}$, so $\overrightarrow{d}(k)$ acts like the BCS gap in determining the excitation spectrum. This has important ramifications for persistent angular momentum.

The two most important pairing states in supefluid $^3He$ are,
1) The Anderson-Brinkman-Morel (ABM) or A-phase. In this case, the order parameter has the form,
$\overrightarrow{d}(k) = (\sqrt{\frac{3}{4\pi}} \sin\theta_k(\cos\phi_k + \sin\phi_k), 0, 0)$, where the angles are the traditional polar angles on the spherical Fermi surface. In this case $|\overrightarrow{d}(k)| \sim \sin\theta_k$ and the gap function goes to zero at the north and south poles of the Fermi surface (point nodes). The order parameter points in the $x$-direction all over the Fermi sphere. Due to the point nodes, many properties show power-law-in-temperature behavior. There is no persistent angular momentum in the A-phase due to the existence of excitations at arbitrarily small energies near the nodes. The wavefunction is made up of the $S_z = \pm 1$ components of the spin singlet. The A1 phase (in the presence of a magnetic field) favors one of these two $S_z$ states.

Why does the A-phase of superfluid $^3He$ not support a persistent current? For example a d-wave superconductor has nodes in the energy gap but it still supports quantized vortices and persistent currents. Also superfluid $^4He$ supports persistent currents. Why don't we have this also for the A-phase? The answer is that s-wave and d-wave superconductors, along with $^4He$, are all described by a complex order parameter of the form $|\psi|e^{i\phi}$. As such one can derive fluxoid and circulating current quantization conditions by demanding that the macroscopic quantum wavefunction be single valued. This puts a quantization constraint on the phase winding number. In $^3He$ A-phase and in p-wave (or f-wave) superconductors the order parameter is a vector and is not simply constrained as in the complex order parameter case. In fact, these systems are much richer and can support a variety of exotic topological defects that are far more interesting than vortices!

2) The Balain-Werthamer (BW) or B-phase. In this case, the order parameter has the form,
$\overrightarrow{d}(k) = \sqrt{\frac{3}{4\pi}}(\sin\theta_k\cos\phi_k, \sin\theta_k\sin\phi_k, \cos\theta_k)$. In this case the order parameter is pointed radially outward on the Fermi surface, and it has a non-zero magnitude everywhere. The system is fully gapped and shows exponentially activated properties at low temperature. The wavefunction is made up of the $S_z = 0$ part of the spin-triplet wavefunction.